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CONTACT DEFORMATION OF AN ELASTIC COMPOSITION OF A HALF-PLANE AND A STRIP OF VARIABLE WIDTH†

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Asymptotic dependences of the deformation on the contact stresses are derived for a strip of variable width bound to an elastic half-plane. Similar relations were previously obtained in [1] for a strip of constant width. © 2001 Elsevier Science Ltd. All rights reserved.

1. RELATIONS FOR AN ELASTIC STRIP OF VARIABLE WIDTH

In the xy coordinate plane, we consider an elastic strip Π of variable width (Fig. 1), the upper boundary Γ_+ of which is described by the function $y = h^+(x)$ and the lower boundary Γ_- is rectilinear: $y = -b$.

Below, we will use a prime on the symbol of a function to denote a derivative with respect to the x coordinate. For brevity, the arguments of functions may be omitted.

Suppose u and v are the displacements along the x and y axes, respectively and that w_k ($k = 1, \ldots, 4$) are the displacements of the boundaries of the strip:

$$u|_{\Gamma_-} = w_1(x), \quad v|_{\Gamma_-} = w_2(x), \quad u|_{\Gamma_+} = w_3(x), \quad v|_{\Gamma_+} = w_4(x)$$

In addition, we will denote the shear and the normal stresses on the upper and lower boundaries of the strip (Fig. 1) by $q_\tau^\pm$ and $q_\nu^\pm$.

According to the results obtained previously [2], the following asymptotic relations hold in the case of a thin strip with a gradual by changing width

$$\begin{aligned}
\frac{1}{2G}q_\tau^+(x) &= \frac{1}{2h(x)}(w_3(x)-w_1(x)) + \frac{æh'(x)}{2(æ-1)h(x)}(w_4(x)-w_2(x)) - \\
&-\frac{æ(2æ+1)}{6(æ^2-1)}h'(x)w_1'(x) + \frac{1}{2(æ-1)}w_2'(x) - \frac{(4æ^2-æ-6)}{6(æ^2-1)}h'(x)w_3'(x) + \\
&+\frac{æ-2}{2(æ-1)}w_4'(x) + A_\tau h(x)w_1''(x) + B_\tau h(x)w_3''(x) \\
\frac{1}{2G}q_\nu^+(x) &= \frac{\beta_*}{2h(x)}(w_4(x)-w_2(x)) - \frac{æh'(x)}{2(æ-1)h(x)}(w_3(x)-w_1(x)) + \\
&+\frac{1}{2(æ-1)}w_1'(x) - \frac{æ(2æ-1)}{6(æ-1)^2}h'(x)w_2'(x) - \frac{æ-2}{2(æ-1)}w_3'(x) - \\
&-\frac{(æ-2)(4æ-3)}{6(æ-1)^2}h'(x)w_4'(x) + A_\nu h(x)w_2''(x) + B_\nu h(x)w_4''(x) \\
\begin{Bmatrix} A_\tau \\ A_\nu \end{Bmatrix} &= \frac{\mp 1}{4(æ-1)}\left[1 \pm \frac{æ(æ\mp 1)}{3(æ\pm 1)}\right], \quad \begin{Bmatrix} B_\tau \\ B_\nu \end{Bmatrix} = \frac{-æ(æ\pm 2)}{6(æ-1)(æ\pm 1)} \\
\beta_* &= (æ+1)(æ-1)^{-1}, \quad h(x) = b + h^+(x), \quad æ = 3-4\nu
\end{aligned} \tag{1.1}$$

where ν is Poisson's ratio and G is the shear modulus.

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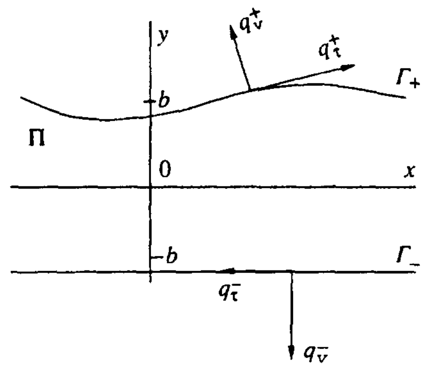

Fig. 1

By using the auxiliary coordinate $\tilde{x} = -x$, $\tilde{y} = -y$ and the corresponding stresses $\tilde{q}_\tau^+ = q_\tau^-$, $\tilde{q}_\nu^+ = q_\nu^-$, the following equalities can be obtained from (1.1)

$$
\begin{aligned}
&\frac{1}{2G}q_\tau^-(x) = \frac{1}{2h(x)}(w_3(x) - w_1(x)) - \frac{h'(x)}{2(æ-1)h(x)}(w_4(x) - w_2(x)) + \\
&+\frac{æ(æ+2)}{6(æ^2-1)}h'(x)w_1'(x) + \frac{æ-2}{2(æ-1)}w_2'(x) - \frac{æ(æ+2)}{6(æ^2-1)}h'(x)w_3'(x) + \\
&+\frac{1}{2(æ-1)}w_4'(x) - A_\tau h(x)w_3''(x) - B_\tau h(x)w_1''(x) \\
&\frac{1}{2G}q_\nu^-(x) = \frac{\beta_*}{2h(x)}(w_4(x) - w_2(x)) - \frac{h'(x)}{2(æ-1)h(x)}(w_3(x) - w_1(x)) - \\
&-\frac{æ-2}{2(æ-1)}w_1'(x) + \frac{æ(æ-2)}{6(æ-1)^2}h'(x)w_2'(x) + \frac{1}{2(æ-1)}w_3'(x) - \\
&-\frac{æ(æ-2)}{6(æ-1)^2}h'(x)w_4'(x) - A_\nu h(x)w_4''(x) - B_\nu h(x)w_2''(x)
\end{aligned} \tag{1.2}
$$

The quantities h, h', h'' will subsequently be assumed to be small quantities of the order of ε, that is $h(x) = \varepsilon\bar{h}(x)$, where ε is a small parameter and $\bar{h}(x)$ is a specified function. In this case, the terms containing a product of the quantities h, h', h'' will be of the second order of smallness and will therefore be omitted.

It immediately follows from (1.1) that the difference $w_3 - w_1$ and $w_4 - w_2$ and, consequently, their derivatives will be of the order of ε. When this fact is taken into account in the result of the differentiation of (1.1), which have been multiplied by h beforehand, we obtain

$$
\begin{aligned}
&\frac{1}{G}(h(x)q_\tau^+(x))' = w_3'(x) - w_1'(x) + (h(x)w_2'(x))' \\
&\frac{æ-1}{G(æ+1)}(h(x)q_\nu^+(x))' = w_4'(x) - w_2'(x) - \frac{æ-3}{æ+1}(h(x)w_1'(x))'
\end{aligned} \tag{1.3}
$$

Relations (1.3) enable us to find expressions for the stresses $q_{\tau,\nu}^-$ in terms of $q_{\tau,\nu}^+$. In order to do this, we subtract the first and second equalities of (1.2) from the corresponding equalities of (1.1) and, from the relations obtained in this way, we express the differences $(w_3' - w_1')$, $(w_4' - w_2')$, taking account of the expressions for $(w_3 - w_1)$, $(w_4 - w_2)$, which hold by virtue of (1.1). As a result we obtain the relations

$$
\begin{aligned}
&w_3'(x) - w_1'(x) = -\frac{1}{G}(q_\nu^+(x) - q_\nu^-(x)) - \frac{1}{G}h'(x)q_\tau^+(x) - (h(x)w_2'(x))' \\
&w_4'(x) - w_2'(x) = \frac{æ-1}{G(æ-3)}(q_\tau^+(x) - q_\tau^-(x)) - \frac{æ-1}{G(æ-3)}h'(x)q_\nu^+(x) +
\end{aligned}
$$

$$+\frac{æ+1}{æ-3}(h(x)w_1'(x))'$$

and, substituting these into (1.3), we arrive at the required expressions

$$q_\tau^+(x)-q_\tau^-(x)=\frac{æ-3}{æ+1}(h(x)q_\nu^+(x))'+h'(x)q_\nu^+(x)-\frac{8G}{æ+1}(h(x)w_1'(x))'$$

$$q_\nu^+(x)-q_\nu^-(x)=-(h(x)q_\tau^+(x))'-h'(x)q_\tau^+(x) \tag{1.4}$$

2. RELATIONS FOR A COMPOSITION CONSISTING OF A STRIP AND A HALF-PLANE

We shall henceforth assume that the boundary Γ_- of the strip II is bound to an elastic half-plane (Fig. 1) so that [3]

$$w_1'(x)=-C_2q_\nu^-(x)+D_2\int_{-\infty}^{\infty}\frac{q_\tau^-(s)}{s-x}ds\equiv-C_2q_\nu^-(x)+D_2K_\tau^-(x)$$

$$w_2'(x)=C_2q_\tau^-(x)+D_2\int_{-\infty}^{\infty}\frac{q_\nu^-(s)}{s-x}ds\equiv C_2q_\tau^-(x)+D_2K_\nu^-(x) \tag{2.1}$$

where $C_2 = -(1-2\nu_2)(2G_2)^{-1}$, $D_2 = (1-\nu_2)(\pi G_2)^{-1}$, ν_2, G_2 are the elastic constants of the half-plane. The subscript 1 is subsequently used to denote the elasticity constants of the strip.

We will assume that the contact stresses $q^\pm_{\tau,\nu}$ are functions with zero values outside a certain contact area (a, b), and that the derivatives of these functions satisfy the Hölder condition in $[a, b]$

$$q^+_{\tau,\nu}(x)=0,\quad x\,\overline{\in}\,(a,b);\quad q^{+'}_{\tau,\nu}(x)\in H[a,b] \tag{2.2}$$

As regards the width of the strip, in addition to the known constrains [2], it is assumed that the corresponding function $h(x)$ and its derivative satisfy the Hölder condition on any segment of the x axis, that is

$$h^{(k)}(x)\in H(-\infty,\infty),\quad k=0,1 \tag{2.3}$$

The aim of the subsequent calculations is to obtain relations between the boundary deformations w_3', w_4' and the contact stresses q_τ^+, q_ν^+. The following scheme will be used here: the quantities w_1', w_2' are eliminated using relations (2.1) and (1.3) after which the stresses q_τ^-, q_ν^- in the resulting equalities are expressed in terms of q_τ^+, q_ν^+ in accordance with (1.4).

In order to carry out these transformations, two assertions are required.

Assertion 1. Suppose $f(x)\in H[a, b]$. Then

$$\frac{d}{dx}\int_a^b\frac{f(s)ds}{s-x}=\int_a^b\frac{f'(s)ds}{s-x}-\left.\frac{f(s)}{s-x}\right|_a^b,\quad x\in(-\infty,\infty) \tag{2.4}$$

Proof. We introduce the function $\psi(x) = f(x) - r(x)$, where

$$r(x)=\frac{1}{b-a}[(x-a)f(b)+(b-x)f(a)]-\frac{(x-a)(b-x)}{(b-a)^2}[(x-a)f_2-(b-x)f_1]$$

$$f_1=f'(a)-\frac{f(b)-f(a)}{b-a},\quad f_2=f'(b)-\frac{f(b)-f(a)}{b-a}$$

so that

$$\psi(a)=\psi(b)=\psi'(a)=\psi'(b)=0,\qquad \psi'(x)\in H[a,b] \tag{2.5}$$

Relations (2.5) enable us, by considering an arbitrary smooth closed contour L containing an arbitrary interval $[\tilde{a}, \tilde{b}] \supset [a, b]$ on the real axis in the complex plane $z = x + iy$ and equating the function $\psi(t)$ at points t of the contour outside the interval $[a, b]$ to zero, to establish that $\psi'(t) \in H(L)$, $t \in L$ and, consequently, [4]

$$\frac{d}{dx}\int_a^b \frac{\psi(t)dt}{t-x} = \int_a^b \frac{\psi'(t)dt}{t-x}, \quad x \in [\tilde{a}, \tilde{b}]$$

Replacing the function $\psi(x)$ in the last equality by the difference $f(x) - r(x)$, taking the integrals of the functions $r(x)$ and $r'(x)$ and taking the arbitrariness in the choice of the interval $[\tilde{a}, \tilde{b}] \supset [a, b]$ into account, we obtain relation (2.4).

Assertion 2. Suppose $f(x) \in H[a, b]$, $g(x) \in H(-\infty, \infty)$ and $f(x) = 0$, $x \bar{\in} (a, b)$. Then

$$\int_{-\infty}^{\infty} \frac{g(s)}{s-x}\left(\int_a^b \frac{f(t)dt}{t-s}\right)ds = -\pi^2 f(x)g(x) + \int_a^b f(t)K(t,x)dt \tag{2.6}$$

where

$$K(t,x) = \int_{-\infty}^{\infty} \frac{\rho(s,x)ds}{(s-x)(t-s)}, \quad \rho(s,x) = g(s) - g(x)$$

Proof. Equality (2.6) is obtained on taking infinite limits of integration in the Poincaré–Bertrand formula [5] and making use of the fact that the function $f(x)$ is equal to zero outside the interval (a, b).

We now substitute (2.1) into (1.3) and, in them, we replace q_τ^-, q_ν^- by q_τ^+, q_ν^+ in accordance with expressions (1.4). In the equalities obtained in this way, we introduce, using Assertion 1, the operation of differentiation under the integral sign and using Assertion 2 and taking account of condition (2.2), we carry out the following transformation of the double integral obtained in the first of these equalities

$$\int_{-\infty}^{\infty} \frac{(h(s)K_\tau^-(s))'}{s-x}ds = -\pi^2(h(x)q_\tau^+(x))' + R(x) + O(\varepsilon^2)$$

where

$$R(x) = \int_a^b q_\tau^{+'}(t)K_0(t,x)dt + \int_a^b q_\tau^+(t)K_1(t,x)dt$$

$$K_m(t,x) = \frac{1}{t-x}\left[\int_{-\infty}^{\infty} \frac{h^{(m)}(s)}{s-x}ds - \int_{-\infty}^{\infty} \frac{h^{(m)}(s)}{s-t}ds\right]$$

As a result, we obtain the required relations

$$\begin{aligned} w_3'(x) = {} & \frac{k_\tau}{G_1}(h(x)q_\tau^+(x))' - C_2(q_\nu^+(x) + h'(x)q_\tau^+(x)) + D_2K_\tau^+(x) - \\ & -D_2\mu h(x)K_\nu^{+'}(x) - D_2h'(x)K_\nu^+(x) + \frac{8G_1D_2^2}{(æ_1+1)}R(x) - \\ & -\frac{2D_2(æ_1-1)}{(æ_1+1)}\left[1 - n\frac{(æ_2-1)}{(æ_1-1)}\right]\int_{-\infty}^{\infty} \frac{h'(s)q_\nu^+(s)}{s-x}ds - D_2(\mu-1)\int_{-\infty}^{\infty} \frac{\rho(s,x)q_\nu^{+'}(s)}{s-x}ds \end{aligned} \tag{2.7}$$

$$\begin{aligned} w_4'(x) = {} & Bk_\nu(h(x)q_\nu^+(x))' + C_2(q_\tau^+(x) - h'(x)q_\nu^+(x)) + D_2K_\nu^+(x) + \\ & + D_2\mu h(x)K_\tau^{+'}(x) + D_2(\mu-1)h'(x)K_\tau^+(x) + 2D_2\int_{-\infty}^{\infty} \frac{h'(s)q_\tau^+(s)}{s-x}ds + D_2\int_{-\infty}^{\infty} \frac{\rho(s,x)q_\tau^{+'}(s)}{s-x}ds \end{aligned}$$

where

$$\rho(s,x)=h(s)-h(x),\quad \mu=\frac{2(æ_2-1)}{(æ_1+1)}\left[\frac{æ_1-1}{æ_2-1}-n\right],\quad n=\frac{G_1}{G_2},\quad B=\frac{(æ_1-1)}{G_1(æ_1+1)}$$

$$k_\tau=1+\frac{n}{2}(æ_2-1)-\frac{n^2}{2}\frac{(æ_2+1)^2}{(æ_1+1)},\quad k_\nu=1+\frac{n}{2}\frac{(æ_1-3)(æ_2-1)}{(æ_1-1)}-\frac{n^2}{2}\frac{(æ_2-1)^2}{(æ_1-1)}$$

Remark. On putting $h(x)$ = const, $q_\tau^+(x)=0$ in (2.7), we arrive at the relations from [1], while for a uniform half-plane, when $h(x)$ = const, $\nu_1=\nu_2$, $G_1=G_2$, equalities (2.7) take the form of (2.1) from [3].

In the case when $\nu_1=\nu_2$, $G_1=G_2$, that is, of a uniform half-plane with an irregular boundary, we have from (2.7)

$$\begin{aligned}
w_3'(x)&=-C_2(q_\nu^+(x)+h'(x)q_\tau^+(x))+D_2K_\tau^+(x)-D_2h'(x)K_\nu^+(x)+\\
&+\frac{8G_2D_2^2}{(æ_1+1)}R(x)+D_2\int_{-\infty}^{\infty}\frac{\rho(s,x)q_\nu^{+\prime}(s)}{s-x}ds\\
w_4'(x)&=C_2(q_\tau^+(x)-h'(x)q_\nu^+(x))+D_2K_\nu^+(x)-D_2h'(x)K_\tau^+(x)+\\
&+2D_2\int_{-\infty}^{\infty}\frac{h'(s)q_\tau^+(s)}{s-x}ds+D_2\int_{-\infty}^{\infty}\frac{\rho(s,x)q_\tau^{+\prime}(s)}{s-x}ds
\end{aligned}\tag{2.8}$$

3. NUMERICAL VERIFICATION OF THE CALCULATIONS

We consider the half-plane $y\leqslant 0$, to the boundary $y=0$ of which the stresses $\tau_{xy}|_{y=0}=\tau(x)$, $\sigma_y|_{y=0}=\sigma(x)$ are applied. Within this half-plane, we draw the curve $\Gamma_+: y=h(x)-h_*$. The displacements of Γ_+ along the x and y axes are denoted by w_3 and w_4 and the shear and normal stresses on Γ_+ are denoted by q_τ^+ and q_ν^+.

The quantities w_3, w_4 and q_τ^+, q_ν^+ can be found starting from the known solution of the boundary-value problem for the half-plane $y\leqslant 0$. On the other hand, these quantities correspond to a problem on the deformation of a half-plane with an irregular boundary Γ_+ and must be associated with relations (2.8). Hence, by determining the quantities w_3, w_4 and q_τ^+, q_ν^+ in Γ_+ from the solution of the problem for the half-plane $y\leqslant 0$ and substituting them into relations (2.8), it is possible to verify these relations.

As an example, we will consider the case when

$$h(x)=\frac{h_*}{1+(x/l)^2},\quad \tau(x)=\begin{cases}\tau, & x\in[-a_0,a_0]\\ 0, & x\overline{\in}[-a_0,a_0]\end{cases},\quad \sigma(x)=\begin{cases}\sigma, & x\in[-a_0,a_0]\\ 0, & x\overline{\in}[-a_0,a_0]\end{cases}\tag{3.1}$$

where $h_*>0$, $l>0$, $a_0>0$, τ, σ are certain parameters. We will check equalities (2.8) at the point $x=0$.

According to definition (3.1), the curve Γ_+ in the neighbourhood of the point $x=0$ coincides with the boundary of the half-plane $y\leqslant 0$. The exact value of $w_3'(0)$ and $w_4'(0)$ can therefore be obtained by making use of the relations for a half-plane of the form of (2.1). As a result, taking account of expression (3.1) for the contact stresses, we find

$$w_3'(0)=-C_2\sigma,\quad w_4'(0)=C_2\tau\tag{3.2}$$

Using the well known formulae [6] for the stresses in a half-plane for the piecewise-constant contact stresses (3.1), we obtain the following expression for the stresses q_τ^+, q_ν^+ in Γ_+ apart from to terms $O(\varepsilon^2)$

$$\begin{aligned}
q_\tau^+(x)&=\frac{2}{\pi}h'(x)[\sigma z^3I_0(x)-\tau z^2I_1(x)-\sigma zI_2(x)+\tau I_3(x)]-\frac{2}{\pi}[\sigma z^2I_1(x)-\tau zI_2(x)]\\
q_\nu^+(x)&=\frac{2}{\pi}[\sigma z^3I_0(x)-\tau z^2I_1(x)]+\frac{4}{\pi}h'(x)[\sigma z^2I_1(x)-\tau zI_2(x)]
\end{aligned}\tag{3.3}$$

in which $z=h_*-h(x)$ and the integrals

$$I_k(x) \equiv \int_a^b \frac{(x-t)^k \, dt}{[(x-t)^2+z^2]^2}$$

are taken in explicit form in [7].

In the case of a function $h(x)$ of the form of (3.1) and $x = 0$, the kernels $K_{0,1}$ from (2.8) have the form

$$K_0(t,0) = \frac{\pi h_*}{l(1+T^2)}, \quad K_1(t,0) = -\frac{\pi h_* T(3+T^2)}{l^2(1+T^2)^2}, \quad T = \frac{t}{l} \tag{3.4}$$

Equalities (3.1), (3.3) and (3.4) define the integrands in (2.8) in explicit form. By carrying out numerical integration, it is possible to find the right-hand sides of (2.8) and to compare them with the values of $w_3'(0)$, $w_4'(0)$ which are known from (3.2).

Calculations were carried out for $l = 1$ m, $a_0 = 1$ m, $\tau = -10^5$ Pa, $\sigma = 10^5$ Pa, $\nu_2 = 0.25$ and $G_2 = 10^{10}$ Pa.

Graphs of the calculated values of the right-hand sides of the first and second relations of (2.8) when $x = 0$ on h_* are shown in Fig. 2 (the solid lines). The values $w_3'(0)$ and $w_4'(0)$ of the left-hand sides of (2.8), according to (3.2), are represented by the dot-dash lines. The dashed lines 1 correspond to the right-hand sides of (2.8) with the last terms omitted, while the dashed lines 2 correspond to the classical expressions for $w_3'(0)$, $w_4'(0)$ of the form (2.1).

These graphs indicate that the deviation of the left-hand and right-hand sides of (2.8) is reduced to zero when $h_* \to 0$ and that the deviation is of the second order of smallness with respect to h_*. This is evidence of the validity of relations (2.8), the accuracy of which is also of the second order of smallness with respect to $\varepsilon \sim h_*$ (we recall that the terms $O(\varepsilon^2)$ were omitted in the derivation of (2.7) and (2.8)).

The retention of certain terms on the right-hand sides of relations (2.8) decreases the order of their accuracy. The classical expressions of the form (2.1) also do not have such a high accuracy as (2.8).

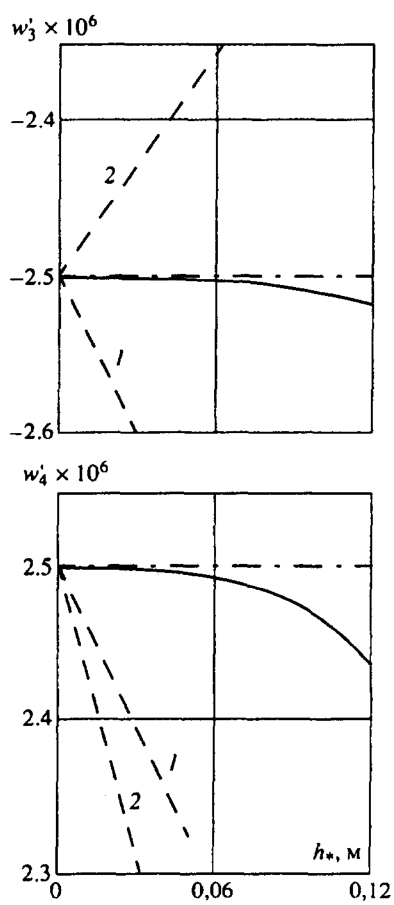

Fig. 2

4. EXAMPLES

Relations (2.7) extend the well-known relation [1] to the case of a strip with a gradually changing width. A question arises in connection with this: how substantial can the difference in the results turn out to be when relations (2.7) are used instead of those obtained previously in [1]? In order to answer this question, two examples are considered below which involve an analysis of the wear of a thin coating, the deformation properties of which are described by the model of an elastic strip which has been considered above.

Consider a coating of thickness h_0 = const bonded to a rigid base into which a punch of cylindrical shape with the generatrix perpendicular to the xy plane (Fig. 1) and moving at a velocity V_s is pressed. As a result of this interaction, the coating is worn away and its thickness decreases: $h = h_0 - W$ and, furthermore, the rate of wear W is assumed to be proportional to the contact pressure $p = -q_y^+$:

$$\frac{\partial W(x,t)}{\partial t} \equiv -\frac{\partial h(x,t)}{\partial t} = \alpha_0 \mid V_s \mid p(x,t), \quad W(x,0) = 0, \quad h(x,0) = h_0 \tag{4.1}$$

To determine the displacement along the y axis of the upper boundary of the coating we make use of the equality $w_4 = -B\tilde{h}p$ which, when $\tilde{h} = h$, follows from the second relation of (2.7) and, when $\tilde{h} = h_0$, follows from the relations obtained in [1], if account is taken of the fact that $G_2 = \infty$ in the case of a rigid base. Then, the condition for contact between the punch and the coating is represented in the form [8, 9]

$$B\tilde{h}(x,t)p(x,t) + W(x,t) = d(x,t) \tag{4.2}$$

$$d(x,t) = -g(x) + \delta(t), \quad \tilde{h}(x,t) = \{h(x,t), h_0\}$$

where $g(x)$ is the shape of the punch and $\delta(t)$ is its penetration.

Equations (4.1) and (4.2) describe the process of wear of a coating on a rigid base and the case $\tilde{h}(x,t) = h(x,t)$ corresponds to relations (2.7) while, when $\tilde{h}(x,t) = h_0$, the relations in [1] are used and the elastic compliance of the coating during its wearing is assumed to be invariable [8].

1. Suppose the punch moves in a perpendicular direction to the xy plane and that its area of contact with the coating is constant (a similar formulation of a contact wear problem was considered, for example, in [9]). We will assume that the penetration δ increases linearly with time: $\delta(t) = \delta_0 + \delta_1 t$. On eliminating the function $p(x,t)$ from Eq. (4.2) using (4.1) and employing the notation $\alpha = \alpha_0 \mid V_s \mid$, we obtain the differential equation

$$\frac{B}{\alpha}\tilde{h}(x,t)\frac{\partial h(x,t)}{\partial t} + h(x,t) = h_0 + g(x) - \delta_0 - \delta_1 t, \quad h(x,0) = h_0 \tag{4.3}$$

the solution of which, when $\tilde{h}(x,t) = h_0$, is obtained in an elementary manner and, when $\tilde{h}(x,t) = h(x,t)$, it can be found in parametric form [10] (ξ is the parameter).

$$h(x,t) = h_0 \frac{A_2(x)}{A_1} \xi \exp[\varphi(\xi_0) - \varphi(\xi)]$$

$$t = \frac{Bh_0}{\alpha A_1} A_2(x)\{1 - \exp[\varphi(\xi_0) - \varphi(\xi)]\} \tag{4.4}$$

$$\xi_0 = \frac{A_1}{A_2(x)}, \quad A_1 = \frac{1}{\alpha}\delta_1 B, \quad A_2(x) = \frac{1}{h_0}[h_0 - \delta_0 + g(x)]$$

$$\exp[\varphi(\xi)] = \mid \xi - \xi_1 \mid^{(1-\gamma)/2} \mid \xi - \xi_2 \mid^{(1+\gamma)/2}, \quad \xi_{1,2} = \frac{1}{2}[1 \mp (1 - 4A_1)^{1/2}]$$

According to Eqs (4.4), when $g(x) \equiv 0$, the contact pressure p and the thickness h solely depend on the time t, being constants within the limits of the contact area. A graph of $p(t)$ is shown in Fig. 3. The dashed curve corresponds to the solution of Eq. (4.3) when $\tilde{h}(x,t) = h_0$.

2. Suppose a parabolic punch moves in the xy plane with a velocity V_s in the opposite direction to the x axis. We associate a system of coordinates with the punch and consider the steady wearing of the coating in this system (a similar formulation of the problem was considered in [11], for example).

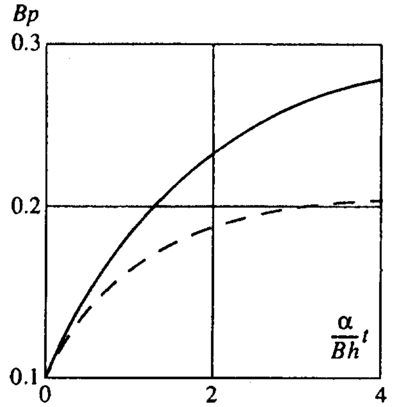

Fig. 3

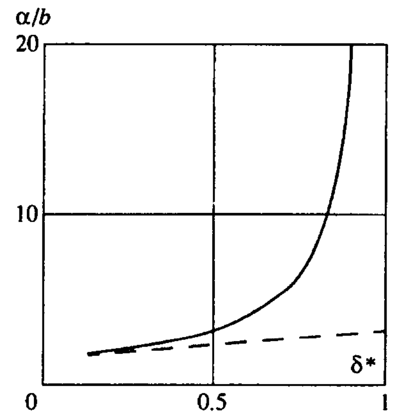

Fig. 4

If the contact area is denoted by $[-a, b]$, then, in the case of a parabolic punch $g(x) = kx^2$, the right-hand side of (4.2) has the form $d(x) = k(a^2 - x^2)$. Moreover, the relation $W'(x) = -h'(x) = \alpha_0 p(x)$ [11] holds.

Using the last two equalities from (4.2), it is possible to obtain the equation

$$\frac{B}{\alpha_0}\tilde{h}(x)h'(x) + h(x) = h_0 - k(a^2 - x^2) \tag{4.5}$$

subject to the following conditions: $h(-a) = h_0, p(-a) = p(b) = 0$. The corresponding solution can be constructed numerically without difficulty using the difference scheme in [12].

Graphs of the ratio a/b, which characterizes the asymmetry of the contact area, against the magnitude of the relative penetration $\delta^* = ka^2/h_0$, are presented in Fig. 4. The solid curve is obtained from Eq. (4.5) when $\tilde{h}(x) = h(x)$ and the dashed curve is obtained when $\tilde{h}(x) = h_0$. The magnitude of $|h'(x)|$ did not exceed 5×10^{-3} for the graphs shown.

The graphs shown in Figs 3 and 4 indicate that allowing for the change in the elastic compliance of the coating on the basis of relations (2.7) can have a substantial effect on the results of the calculation of the wear of a coating compared with what takes place when the relations previously obtained in [1] are used.

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